

CHAPTER 3

Limits and Continuity of Functions

The notions of limits and continuity of functions lie at the kernel of calculus. The general concept of continuity is very old in mathematics. It had its inception long ago in ancient Greece. We owe to Aristotle (384–322 B.C.) the first known definition of continuity: “A thing is continuous when of any two successive parts the limits at which they touch are one and the same and are, as the word implies, held together” (see Smith, 1958, page 93). Our present definitions of limits and continuity of functions, however, are substantially those given by Augustin-Louis Cauchy (1789–1857).

In this chapter we introduce the concepts of limits and continuity of real-valued functions, and study some of their properties. The domains of definition of the functions will be subsets of R , the set of real numbers. A typical subset of R will be denoted by D .

3.1. LIMITS OF A FUNCTION

Before defining the notion of a limit of a function, let us understand what is meant by the notation $x \rightarrow a$, where a and x are elements in R . If a is finite, then $x \rightarrow a$ means that x can have values that belong to a neighborhood $N_r(a)$ of a (see Definition 1.6.1) for any $r > 0$, but $x \neq a$, that is, $0 < |x - a| < r$. Such a neighborhood is called a deleted neighborhood of a , that is, a neighborhood from which the point a has been removed. If a is infinite ($-\infty$ or $+\infty$), then $x \rightarrow a$ indicates that $|x|$ can get larger and larger without any constraint on the extent of its increase. Thus $|x|$ can have values greater than any positive number. In either case, whether a is finite or infinite, we say that x tends to a or approaches a .

Let us now study the behavior of a function $f(x)$ as $x \rightarrow a$.

Definition 3.1.1. Suppose that the function $f(x)$ is defined in a deleted neighborhood of a point $a \in R$. Then $f(x)$ is said to have a limit L as $x \rightarrow a$

if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad (3.1)$$

for all x for which

$$0 < |x - a| < \delta. \quad (3.2)$$

In this case, we write $f(x) \rightarrow L$ as $x \rightarrow a$, which is equivalent to saying that $\lim_{x \rightarrow a} f(x) = L$. Less formally, we say that $f(x) \rightarrow L$ as $x \rightarrow a$ if, however small the positive number ϵ might be, $f(x)$ differs from L by less than ϵ for values of x sufficiently close to a . $\square$

NOTE 3.1.1. When $f(x)$ has a limit as $x \rightarrow a$, it is considered to be finite. If this is not the case, then $f(x)$ is said to have an infinite limit ($-\infty$ or $+\infty$) as $x \rightarrow a$. This limit exists only in the extended real number system, which consists of the real number system combined with the two symbols, $-\infty$ and $+\infty$. In this case, for every positive number M there exists a $\delta > 0$ such that $|f(x)| > M$ if $0 < |x - a| < \delta$. If a is infinite and L is finite, then $f(x) \rightarrow L$ as $x \rightarrow a$ if for any $\epsilon > 0$ there exists a positive number N such that inequality (3.1) is satisfied for all x for which $|x| > N$. In case both a and L are infinite, then $f(x) \rightarrow L$ as $x \rightarrow a$ if for any $B > 0$ there exists a positive number A such that $|f(x)| > B$ if $|x| > A$.

NOTE 3.1.2. If $f(x)$ has a limit L as $x \rightarrow a$, then L must be unique. To show this, suppose that L_1 and L_2 are two limits of $f(x)$ as $x \rightarrow a$. Then, for any $\epsilon > 0$ there exist $\delta_1 > 0$, $\delta_2 > 0$ such that

$$\begin{aligned} |f(x) - L_1| &< \frac{\epsilon}{2}, & \text{if } 0 < |x - a| < \delta_1, \\ |f(x) - L_2| &< \frac{\epsilon}{2}, & \text{if } 0 < |x - a| < \delta_2. \end{aligned}$$

Hence, if $\delta = \min(\delta_1, \delta_2)$, then

$$\begin{aligned} |L_1 - L_2| &= |L_1 - f(x) + f(x) - L_2| \\ &\leq |f(x) - L_1| + |f(x) - L_2| \\ &< \epsilon \end{aligned}$$

for all x for which $0 < |x - a| < \delta$. Since $|L_1 - L_2|$ is smaller than ϵ , which is an arbitrary positive number, we must have $L_1 = L_2$ (why?).

NOTE 3.1.3. The limit of $f(x)$ as described in Definition 3.1.1 is actually called a two-sided limit. This is because x can approach a from either side. There are, however, cases where $f(x)$ can have a limit only when x approaches a from one side. Such a limit is called a one-sided limit.

By definition, if $f(x)$ has a limit as x approaches a from the left, symbolically written as $x \rightarrow a^-$, then $f(x)$ has a left-sided limit, which we denote by L^- . In this case we write

$$\lim_{x \rightarrow a^-} f(x) = L^-.$$

If, however, $f(x)$ has a limit as x approaches a from the right, symbolically written as $x \rightarrow a^+$, then $f(x)$ has a right-sided limit, denoted by L^+ , that is,

$$\lim_{x \rightarrow a^+} f(x) = L^+.$$

From the above definition it follows that $f(x)$ has a left-sided limit L^- as $x \rightarrow a^-$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - L^-| < \epsilon$$

for all x for which $0 < a - x < \delta$. Similarly, $f(x)$ has a right-sided limit L^+ as $x \rightarrow a^+$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - L^+| < \epsilon$$

for all x for which $0 < x - a < \delta$.

Obviously, if $f(x)$ has a two-sided limit L as $x \rightarrow a$, then L^- and L^+ both exist and are equal to L . Vice versa, if $L^- = L^+$, then $f(x)$ has a two-sided limit L as $x \rightarrow a$, where L is the common value of L^- and L^+ (why?). We can then state that $\lim_{x \rightarrow a} f(x) = L$ if and only if

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = L.$$

Thus to determine if $f(x)$ has a limit as $x \rightarrow a$, we first need to find out if it has a left-sided limit L^- and a right-sided limit L^+ as $x \rightarrow a$. If this is the case and $L^- = L^+ = L$, then $f(x)$ has a limit L as $x \rightarrow a$.

Throughout the remainder of the book, we shall drop the characterization “two-sided” when making a reference to a two-sided limit L of $f(x)$. Instead, we shall simply state that L is the limit of $f(x)$.

EXAMPLE 3.1.1. Consider the function

$$f(x) = \begin{cases} (x-1)/(x^2-1), & x \neq -1, 1, \\ 4, & x = 1. \end{cases}$$

This function is defined everywhere except at $x = -1$. Let us find its limit as $x \rightarrow a$, where $a \in \mathbb{R}$. We note that

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} \frac{x-1}{x^2-1} = \lim_{x \rightarrow a} \frac{1}{x+1}.$$

This is true even if $a = 1$, because $x \neq a$ as $x \rightarrow a$. We now claim that if $a \neq -1$, then

$$\lim_{x \rightarrow a} \frac{1}{x+1} = \frac{1}{a+1}.$$

To prove this claim, we need to find a $\delta > 0$ such that for any $\epsilon > 0$,

$$\left| \frac{1}{x+1} - \frac{1}{a+1} \right| < \epsilon \quad (3.3)$$

if $0 < |x - a| < \delta$. Let us therefore consider the following two cases:

CASE 1. $a > -1$. In this case we have

$$\left| \frac{1}{x+1} - \frac{1}{a+1} \right| = \frac{|x-a|}{|x+1| |a+1|}. \quad (3.4)$$

If $|x - a| < \delta$, then

$$a - \delta + 1 < x + 1 < a + \delta + 1. \quad (3.5)$$

Since $a + 1 > 0$, we can choose $\delta > 0$ such that $a - \delta + 1 > 0$, that is, $\delta < a + 1$. From (3.4) and (3.5) we then get

$$\left| \frac{1}{x+1} - \frac{1}{a+1} \right| < \frac{\delta}{(a+1)(a-\delta+1)}.$$

Let us constrain δ even further by requiring that

$$\frac{\delta}{(a+1)(a-\delta+1)} < \epsilon.$$

This is accomplished by choosing $\delta > 0$ so that

$$\delta < \frac{(a+1)^2 \epsilon}{1 + (a+1) \epsilon}.$$

Since

$$\frac{(a+1)^2 \epsilon}{1 + (a+1) \epsilon} < a + 1,$$

inequality (3.3) will be satisfied by all x for which $|x - a| < \delta$, where

$$0 < \delta < \frac{(a+1)^2 \epsilon}{1 + (a+1) \epsilon}. \quad (3.6)$$

CASE 2. $a < -1$. Here, we choose $\delta > 0$ such that $a + \delta + 1 < 0$, that is, $\delta < -(a + 1)$. From (3.5) we conclude that

$$|x + 1| > -(a + \delta + 1).$$

Hence, from (3.4) we get

$$\left| \frac{1}{x+1} - \frac{1}{a+1} \right| < \frac{\delta}{(a+1)(a+\delta+1)}.$$

As before, we further constrain δ by requiring that it satisfy the inequality

$$\frac{\delta}{(a+1)(a+\delta+1)} < \epsilon,$$

or equivalently, the inequality

$$\delta < \frac{(a+1)^2 \epsilon}{1 - (a+1) \epsilon}.$$

Note that

$$\frac{(a+1)^2 \epsilon}{1 - (a+1) \epsilon} < -(a+1).$$

Consequently, inequality (3.3) can be satisfied by choosing δ such that

$$0 < \delta < \frac{(a+1)^2 \epsilon}{1 - (a+1) \epsilon}. \quad (3.7)$$

Cases 1 and 2 can be combined by rewriting (3.6) and (3.7) using the single double inequality

$$0 < \delta < \frac{|a+1|^2 \epsilon}{1 + |a+1| \epsilon}.$$

If $a = -1$, then no limit exists as $x \rightarrow a$. This is because

$$\lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow -1} \frac{1}{x+1}.$$

If $x \rightarrow -1^-$, then

$$\lim_{x \rightarrow -1^-} \frac{1}{x+1} = -\infty,$$

and, as $x \rightarrow -1^+$,

$$\lim_{x \rightarrow -1^+} \frac{1}{x+1} = \infty.$$

Since the left-sided and right-sided limits are not equal, no limit exists as $x \rightarrow -1$.

EXAMPLE 3.1.2. Let $f(x)$ be defined as

$$f(x) = \begin{cases} 1 + \sqrt{x}, & x \geq 0, \\ x, & x < 0. \end{cases}$$

This function has no limit as $x \rightarrow 0$, since

$$\begin{aligned} \lim_{x \rightarrow 0^-} f(x) &= \lim_{x \rightarrow 0^-} x = 0, \\ \lim_{x \rightarrow 0^+} f(x) &= \lim_{x \rightarrow 0^+} (1 + \sqrt{x}) = 1. \end{aligned}$$

However, for any $a \neq 0$, $\lim_{x \rightarrow a} f(x)$ exists.

EXAMPLE 3.1.3. Let $f(x)$ be given by

$$f(x) = \begin{cases} x \cos x, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

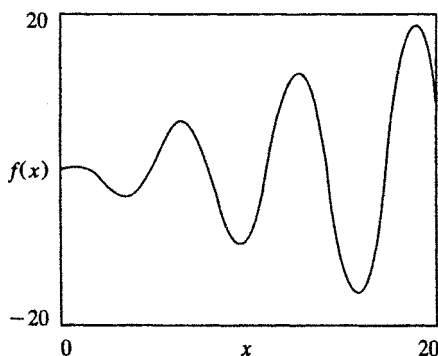


Figure 3.1. The graph of the function $f(x)$.

Then $\lim_{x \rightarrow 0} f(x) = 0$. This is true because $|f(x)| \leq |x|$ in any deleted neighborhood of $a = 0$. As $x \rightarrow \infty$, $f(x)$ oscillates unboundedly, since

$$-x \leq x \cos x \leq x.$$

Thus $f(x)$ has no limit as $x \rightarrow \infty$. A similar conclusion can be reached when $x \rightarrow -\infty$ (see Figure 3.1).

3.2. SOME PROPERTIES ASSOCIATED WITH LIMITS OF FUNCTIONS

The following theorems give some fundamental properties associated with function limits.

Theorem 3.2.1. Let $f(x)$ and $g(x)$ be real-valued functions defined on $D \subset \mathbb{R}$. Suppose that $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$. Then

1. $\lim_{x \rightarrow a} [f(x) + g(x)] = L + M$,
2. $\lim_{x \rightarrow a} [f(x)g(x)] = LM$,
3. $\lim_{x \rightarrow a} [1/g(x)] = 1/M$ if $M \neq 0$,
4. $\lim_{x \rightarrow a} [f(x)/g(x)] = L/M$ if $M \neq 0$.

Proof. We shall only prove parts 2 and 3. The proof of part 1 is straightforward, and part 4 results from applying parts 2 and 3.

Proof of Part 2. Consider the following three cases:

CASE 1. Both L and M are finite. Let $\epsilon > 0$ be given and let $\tau > 0$ be such that

$$\tau(\tau + |L| + |M|) < \epsilon. \quad (3.8)$$

This inequality is satisfied by all values of τ for which

$$0 < \tau < \frac{-(|L| + |M|) + \sqrt{(|L| + |M|)^2 + 4\epsilon}}{2}.$$

Now, there exist $\delta_1 > \delta_2 > 0$ such that

$$\begin{aligned} |f(x) - L| &< \tau & \text{if } 0 < |x - a| < \delta_1, \\ |g(x) - M| &< \tau & \text{if } 0 < |x - a| < \delta_2. \end{aligned}$$

Then, for any x such that $0 < |x - a| < \delta$ where $\delta = \min(\delta_1, \delta_2)$,

$$\begin{aligned}
 |f(x)g(x) - LM| &= |M[f(x) - L] + f(x)[g(x) - M]| \\
 &\leq \tau|M| + \tau|f(x)| \\
 &\leq \tau|M| + \tau[|L| + |f(x) - L|] \\
 &\leq \tau(\tau + |L| + |M|) \\
 &< \epsilon,
 \end{aligned}$$

which proves part 2.

CASE 2. One of L and M is finite and the other is infinite. Without any loss of generality, we assume that L is finite and $M = \infty$. Let us also assume that $L \neq 0$, since $0 \cdot \infty$ is indeterminate. Let $A > 0$ be given. There exists a $\delta_1 > 0$ such that $|f(x)| > |L|/2$ if $0 < |x - a| < \delta_1$ (why?). Also, there exists a $\delta_2 > 0$ such that $|g(x)| > 2A/|L|$ if $0 < |x - a| < \delta_2$. Let $\delta = \min(\delta_1, \delta_2)$. If $0 < |x - a| < \delta$, then

$$\begin{aligned}
 |f(x)g(x)| &= |f(x)||g(x)| \\
 &< \frac{|L|}{2} \frac{2}{|L|} A = A.
 \end{aligned}$$

This means that $\lim_{x \rightarrow a} f(x)g(x) = \infty$, which proves part 2.

CASE 3. Both L and M are infinite. Suppose that $L = \infty, M = \infty$. In this case, for a given $B > 0$ there exist $\kappa_1 > 0, \kappa_2 > 0$ such that

$$\begin{aligned}
 |f(x)| &> \sqrt{B} & \text{if } 0 < |x - a| < \kappa_1, \\
 |g(x)| &> \sqrt{B} & \text{if } 0 < |x - a| < \kappa_2.
 \end{aligned}$$

Then,

$$|f(x)g(x)| > B, \quad \text{if } 0 < |x - a| < \kappa,$$

where $\kappa = \min(\kappa_1, \kappa_2)$. This implies that $\lim_{x \rightarrow a} f(x)g(x) = \infty$, which proves part 2.

Proof of Part 3. Let $\epsilon > 0$ be given. If $M \neq 0$, then there exists a $\lambda_1 > 0$ such that $|g(x)| > |M|/2$ if $0 < |x - a| < \lambda_1$. Also, there exists a $\lambda_2 > 0$ such that $|g(x) - M| < \epsilon M^2/2$ if $0 < |x - a| < \lambda_2$. Then,

$$\begin{aligned}
 \left| \frac{1}{g(x)} - \frac{1}{M} \right| &= \frac{|g(x) - M|}{|g(x)||M|} \\
 &< \frac{2|g(x) - M|}{|M|^2} \\
 &< \epsilon,
 \end{aligned}$$

if $0 < |x - a| < \lambda$, where $\lambda = \min(\lambda_1, \lambda_2)$. $\square$

Theorem 3.2.1 is also true if L and M are one-sided limits of $f(x)$ and $g(x)$, respectively.

Theorem 3.2.2. If $f(x) \leq g(x)$, then $\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x)$.

Proof. Let $\lim_{x \rightarrow a} f(x) = L$, $\lim_{x \rightarrow a} g(x) = M$. Suppose that $L - M > 0$. By Theorem 3.2.1, $L - M$ is the limit of the function $h(x) = f(x) - g(x)$. Therefore, there exists a $\delta > 0$ such that

$$|h(x) - (L - M)| < \frac{L - M}{2}, \quad (3.9)$$

if $0 < |x - a| < \delta$. Inequality (3.9) implies that $h(x) > (L - M)/2 > 0$, which is not possible, since, by assumption, $h(x) = f(x) - g(x) \leq 0$. We must then have $L - M \leq 0$. $\square$

3.3. THE o , O NOTATION

These symbols provide a convenient way to describe the limiting behavior of a function $f(x)$ as x tends to a certain limit.

Let $f(x)$ and $g(x)$ be two functions defined on $D \subset \mathbb{R}$. The function $g(x)$ is positive and usually has a simple form such as 1, x , or $1/x$. Suppose that there exists a positive number K such that

$$|f(x)| \leq Kg(x)$$

for all $x \in E$, where $E \subset D$. Then, $f(x)$ is said to be of an order of magnitude not exceeding that of $g(x)$. This fact is denoted by writing

$$f(x) = O(g(x))$$

for all $x \in E$. In particular, if $g(x) = 1$, then $f(x)$ is necessarily a bounded function on E . For example,

$$\cos x = O(1) \quad \text{for all } x,$$

$$x = O(x^2) \quad \text{for large values of } x,$$

$$x^2 + 2x = O(x^2) \quad \text{for large values of } x,$$

$$\sin x = O(|x|) \quad \text{for all } x.$$

The last relationship is true because

$$\left| \frac{\sin x}{x} \right| \leq 1$$

for all values of x , where x is measured in radians.

Let us now suppose that the relationship between $f(x)$ and $g(x)$ is such that

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = 0.$$

Then we say that $f(x)$ is of a smaller order of magnitude than $g(x)$ in a deleted neighborhood of a . This fact is denoted by writing

$$f(x) = o(g(x)) \quad \text{as } x \rightarrow a,$$

which is equivalent to saying that $f(x)$ tends to zero more rapidly than $g(x)$ as $x \rightarrow a$. The o symbol can also be used when x tends to infinity. In this case we write

$$f(x) = o(g(x)) \quad \text{for } x > A,$$

where A is some positive number. For example,

$$\begin{aligned} x^2 &= o(x) & \text{as } x \rightarrow 0, \\ \tan x^3 &= o(x^2) & \text{as } x \rightarrow 0, \\ \sqrt{x} &= o(x) & \text{as } x \rightarrow \infty. \end{aligned}$$

If $f(x)$ and $g(x)$ are any two functions such that

$$\frac{f(x)}{g(x)} \rightarrow 1 \quad \text{as } x \rightarrow a,$$

then $f(x)$ and $g(x)$ are said to be asymptotically equal, written symbolically $f(x) \sim g(x)$, as $x \rightarrow a$. For example,

$$\begin{aligned} x^2 &\sim x^2 + 3x + 1 & \text{as } x \rightarrow \infty, \\ \sin x &\sim x & \text{as } x \rightarrow 0. \end{aligned}$$

On the basis of the above definitions, the following properties can be deduced:

1. $O(f(x) + g(x)) = O(f(x)) + O(g(x))$.
2. $O(f(x)g(x)) = O(f(x))O(g(x))$.
3. $o(f(x)g(x)) = O(f(x))o(g(x))$.
4. If $f(x) \sim g(x)$ as $x \rightarrow a$, then $f(x) = g(x) + o(g(x))$ as $x \rightarrow a$.

3.4. CONTINUOUS FUNCTIONS

A function $f(x)$ may have a limit L as $x \rightarrow a$. This limit, however, may not be equal to the value of the function at $x = a$. In fact, the function may not even

be defined at this point. If $f(x)$ is defined at $x = a$ and $L = f(a)$, then $f(x)$ is said to be continuous at $x = a$.

Definition 3.4.1. Let $f: D \rightarrow R$, where $D \subset R$, and let $a \in D$. Then $f(x)$ is continuous at $x = a$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - f(a)| < \epsilon$$

for all $x \in D$ for which $|x - a| < \delta$.

It is important here to note that in order for $f(x)$ to be continuous at $x = a$, it is necessary that it be defined at $x = a$ as well as at all other points inside a neighborhood $N_r(a)$ of the point a for some $r > 0$. Thus to show continuity of $f(x)$ at $x = a$, the following conditions must be verified:

1. $f(x)$ is defined at all points inside a neighborhood of the point a .
2. $f(x)$ has a limit from the left and a limit from the right as $x \rightarrow a$, and that these two limits are equal to L .
3. The value of $f(x)$ at $x = a$ is equal to L .

For convenience, we shall denote the left-sided and right-sided limits of $f(x)$ as $x \rightarrow a$ by $f(a^-)$ and $f(a^+)$, respectively.

If any of the above conditions is violated, then $f(x)$ is said to be discontinuous at $x = a$. There are two kinds of discontinuity. $\square$

Definition 3.4.2. A function $f: D \rightarrow R$ has a discontinuity of the first kind at $x = a$ if $f(a^-)$ and $f(a^+)$ exist, but at least one of them is different from $f(a)$. The function $f(x)$ has a discontinuity of the second kind at the same point if at least one of $f(a^-)$ and $f(a^+)$ does not exist. $\square$

Definition 3.4.3. A function $f: D \rightarrow R$ is continuous on $E \subset D$ if it is continuous at every point of E .

For example, the function

$$f(x) = \begin{cases} \frac{x-1}{x^2-1}, & x \geq 0, x \neq 1, \\ \frac{1}{2}, & x = 1, \end{cases}$$

is defined for all $x \geq 0$ and is continuous at $x = 1$. This is true because, as was shown in Example 3.1.1,

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} \frac{1}{x+1} = \frac{1}{2},$$

which is equal to the value of the function at $x = 1$. Furthermore, $f(x)$ is

continuous at all other points of its domain. Note that if $f(1)$ were different from $\frac{1}{2}$, then $f(x)$ would have a discontinuity of the first kind at $x = 1$.

Let us now consider the function

$$f(x) = \begin{cases} x + 1, & x > 0, \\ 0, & x = 0, \\ x - 1, & x < 0. \end{cases}$$

This function is continuous everywhere except at $x = 0$, since it has no limit as $x \rightarrow 0$ by the fact that $f(0^-) = -1$ and $f(0^+) = 1$. The discontinuity at this point is therefore of the first kind.

An example of a discontinuity of the second kind is given by the function

$$f(x) = \begin{cases} \cos \frac{1}{x}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

which has a discontinuity of the second kind at $x = 0$, since neither $f(0^-)$ nor $f(0^+)$ exists.

Definition 3.4.4. A function $f: D \rightarrow R$ is left-continuous at $x = a$ if $\lim_{x \rightarrow a^-} f(x) = f(a)$. It is right-continuous at $x = a$ if $\lim_{x \rightarrow a^+} f(x) = f(a)$. □

Obviously, a left-continuous or a right-continuous function is not necessarily continuous. In order for $f(x)$ to be continuous at $x = a$ it is necessary and sufficient that $f(x)$ be both left-continuous and right-continuous at this point.

For example, the function

$$f(x) = \begin{cases} x - 1, & x \leq 0, \\ 1, & x > 0 \end{cases}$$

is left-continuous at $x = 0$, since $f(0^-) = -1 = f(0)$. If $f(x)$ were defined so that $f(x) = x - 1$ for $x < 0$ and $f(x) = 1$ for $x \geq 0$, then it would be right-continuous at $x = 0$.

Definition 3.4.5. The function $f: D \rightarrow R$ is uniformly continuous on $E \subset D$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x_1) - f(x_2)| < \epsilon \tag{3.10}$$

for all $x_1, x_2 \in E$ for which $|x_1 - x_2| < \delta$. □

This definition appears to be identical to the definition of continuity. That is not exactly the case. Uniform continuity is always associated with a set such

as E in Definition 3.4.5, whereas continuity can be defined at a single point a . Furthermore, inequality (3.10) is true for all pairs of points $x_1, x_2 \in E$ such that $|x_1 - x_2| < \delta$. Hence, δ depends only on ϵ , not on the particular locations of x_1, x_2 . On the other hand, in the definition of continuity (Definition 3.4.1) δ depends on ϵ as well as on the location of the point where continuity is considered. In other words, δ can change from one point to another for the same given $\epsilon > 0$. If, however, for a given $\epsilon > 0$, the same δ can be used with all points in some set $E \subset D$, then $f(x)$ is uniformly continuous on E . For this reason, whenever $f(x)$ is uniformly continuous on E , δ can be described as being “portable,” which means it can be used everywhere inside E provided that $\epsilon > 0$ remains unchanged.

Obviously, if $f(x)$ is uniformly continuous on E , then it is continuous there. The converse, however, is not true. For example, consider the function $f: (0, 1) \rightarrow \mathbb{R}$ given by $f(x) = 1/x$. Here, $f(x)$ is continuous at all points of $E = (0, 1)$, but is not uniformly continuous there. To demonstrate this fact, let us first show that $f(x)$ is continuous on E . Let $\epsilon > 0$ be given and let $a \in E$. Since $a > 0$, there exists a $\delta_1 > 0$ such that the neighborhood $N_{\delta_1}(a)$ is a subset of E . This can be accomplished by choosing δ_1 such that $0 < \delta_1 < a$. Now, for all $x \in N_{\delta_1}(a)$,

$$\left| \frac{1}{x} - \frac{1}{a} \right| = \frac{|x - a|}{ax} < \frac{\delta_1}{a(a - \delta_1)}.$$

Let $\delta_2 > 0$ be such that for the given $\epsilon > 0$,

$$\frac{\delta_2}{a(a - \delta_2)} < \epsilon,$$

which can be satisfied by requiring that

$$0 < \delta_2 < \frac{a^2 \epsilon}{1 + a \epsilon}.$$

Since

$$\frac{a^2 \epsilon}{1 + a \epsilon} < a,$$

then

$$\left| \frac{1}{x} - \frac{1}{a} \right| < \epsilon \tag{3.11}$$

if $|x - a| < \delta$, where

$$\delta < \min\left(a, \frac{a^2\epsilon}{1 + a\epsilon}\right) = \frac{a^2\epsilon}{1 + a\epsilon}.$$

It follows that $f(x) = 1/x$ is continuous at every point of E . We note here the dependence of δ on both ϵ and a .

Let us now demonstrate that $f(x) = 1/x$ is not uniformly continuous on E . Define G to be the set

$$G = \left\{ \frac{a^2\epsilon}{1 + a\epsilon} \mid a \in E \right\}.$$

In order for $f(x) = 1/x$ to be uniformly continuous on E , the infimum of G must be positive. If this is possible, then for a given $\epsilon > 0$, (3.11) will be satisfied by all x for which

$$|x - a| < \inf(G),$$

and for all $a \in (0, 1)$. However, this cannot happen, since $\inf(G) = 0$. Thus it is not possible to find a single δ for which (3.11) will work for all $a \in (0, 1)$.

Let us now try another function defined on the same set $E = (0, 1)$, namely, the function $f(x) = x^2$. In this case, for a given $\epsilon > 0$, let $\delta > 0$ be such that

$$\delta^2 + 2\delta a - \epsilon < 0. \quad (3.12)$$

Then, for any $a \in E$, if $|x - a| < \delta$ we get

$$\begin{aligned} |x^2 - a^2| &= |x - a| |x + a| \\ &= |x - a| |x - a + 2a| \\ &\leq \delta(\delta + 2a) < \epsilon. \end{aligned} \quad (3.13)$$

It is easy to see that this inequality is satisfied by all $\delta > 0$ for which

$$0 < \delta < -a + \sqrt{a^2 + \epsilon}. \quad (3.14)$$

If H is the set

$$H = \{-a + \sqrt{a^2 + \epsilon} \mid a \in E\},$$

then it can be verified that $\inf(H) = -1 + \sqrt{1 + \epsilon}$. Hence, by choosing δ such that

$$\delta \leq \inf(H),$$

inequality (3.14), and hence (3.13), will be satisfied for all $a \in E$. The function $f(x) = x^2$ is therefore uniformly continuous on E .

The above examples demonstrate that continuity and uniform continuity on a set E are not always equivalent. They are, however, equivalent under certain conditions on E . This will be illustrated in Theorem 3.4.6 in the next subsection.

3.4.1. Some Properties of Continuous Functions

Continuous functions have some interesting properties, some of which are given in the following theorems:

Theorem 3.4.1. Let $f(x)$ and $g(x)$ be two continuous functions defined on a set $D \subset \mathbb{R}$. Then:

1. $f(x) + g(x)$ and $f(x)g(x)$ are continuous on D .
2. $af(x)$ is continuous on D , where a is a constant.
3. $f(x)/g(x)$ is continuous on D provided that $g(x) \neq 0$ on D .

Proof. The proof is left as an exercise. □

Theorem 3.4.2. Suppose that $f: D \rightarrow \mathbb{R}$ is continuous on D , and $g: f(D) \rightarrow \mathbb{R}$ is continuous on $f(D)$, the image of D under f . Then the composite function $h: D \rightarrow \mathbb{R}$ defined as $h(x) = g[f(x)]$ is continuous on D .

Proof. Let $\epsilon > 0$ be given, and let $a \in D$. Since g is continuous at $f(a)$, there exists a $\delta' > 0$ such that $|g[f(x)] - g[f(a)]| < \epsilon$ if $|f(x) - f(a)| < \delta'$. Since $f(x)$ is continuous at $x = a$, there exists a $\delta > 0$ such that $|f(x) - f(a)| < \delta'$ if $|x - a| < \delta$. It follows that by taking $|x - a| < \delta$ we must have $|h(x) - h(a)| < \epsilon$. □

Theorem 3.4.3. If $f(x)$ is continuous at $x = a$ and $f(a) > 0$, then there exists a neighborhood $N_\delta(a)$ in which $f(x) > 0$.

Proof. Since $f(x)$ is continuous at $x = a$, there exists a $\delta > 0$ such that

$$|f(x) - f(a)| < \frac{1}{2}f(a),$$

if $|x - a| < \delta$. This implies that

$$f(x) > \frac{1}{2}f(a) > 0$$

for all $x \in N_\delta(a)$. □

Theorem 3.4.4 (The Intermediate-Value Theorem). Let $f: D \rightarrow \mathbb{R}$ be continuous, and let $[a, b]$ be a closed interval contained in D . Suppose that

$f(a) < f(b)$. If λ is a number such that $f(a) < \lambda < f(b)$, then there exists a point c , where $a < c < b$, such that $\lambda = f(c)$.

Proof. Let $g: D \rightarrow R$ be defined as $g(x) = f(x) - \lambda$. This function is continuous and is such that $g(a) < 0, g(b) > 0$. Consider the set

$$S = \{x \in [a, b] \mid g(x) < 0\}.$$

This set is nonempty, since $a \in S$ and is bounded from above by b . Hence, by Theorem 1.5.1 the least upper bound of S exists. Let $c = \text{lub}(S)$. Since $S \subset [a, b]$, then $c \in [a, b]$.

Now, for every positive integer n , there exists a point $x_n \in S$ such that

$$c - \frac{1}{n} < x_n \leq c.$$

Otherwise, if $x \leq c - 1/n$ for all $x \in S$, then $c - 1/n$ will be an upper bound of S , contrary to the definition of c . Consequently, $\lim_{n \rightarrow \infty} x_n = c$. Since $g(x)$ is continuous on $[a, b]$, then

$$g(c) = \lim_{x_n \rightarrow c} g(x_n) \leq 0, \quad (3.15)$$

by Theorem 3.2.2 and the fact that $g(x_n) < 0$. From (3.15) we conclude that $c < b$, since $g(b) > 0$.

Let us suppose that $g(c) < 0$. Then, by Theorem 3.4.3, there exists a neighborhood $N_\delta(c)$, for some $\delta > 0$, such that $g(x) < 0$ for all $x \in N_\delta(c) \cap [a, b]$. Consequently, there exists a point $x_0 \in [a, b]$ such that $c < x_0 < c + \delta$ and $g(x_0) < 0$. This means that x_0 belongs to S and is greater than c , a contradiction. Therefore, by inequality (3.15) we must have $g(c) = 0$, that is, $f(c) = \lambda$. We note that $c > a$, since $c \geq a$, but $c \neq a$. This last is true because if $a = c$, then $g(c) < 0$, a contradiction. This completes the proof of the theorem. $\square$

The direct implication of the intermediate-value theorem is that a continuous function possesses the property of assuming at least once every value between any two distinct values taken inside its domain.

Theorem 3.4.5. Suppose that $f: D \rightarrow R$ is continuous and that D is closed and bounded. Then $f(x)$ is bounded in D .

Proof. Let a be the greatest lower bound of D , which exists because D is bounded. Since D is closed, then $a \in D$ (why?). Furthermore, since $f(x)$

is continuous, then for a given $\epsilon > 0$ there exists a $\delta_1 > 0$ such that

$$f(a) - \epsilon < f(x) < f(a) + \epsilon$$

if $|x - a| < \delta_1$. The function $f(x)$ is therefore bounded in $N_{\delta_1}(a)$. Define $\mathcal{A}$ to be the set

$$\mathcal{A} = \{x \in D \mid f(x) \text{ is bounded}\}.$$

This set is nonempty and bounded, and $N_{\delta_1}(a) \cap D \subset \mathcal{A}$. We need to show that $D - \mathcal{A}$ is an empty set.

As before, the least upper bound of $\mathcal{A}$ exists (since it is bounded) and belongs to D (since D is closed). Let $c = \text{lub}(\mathcal{A})$. By the continuity of $f(x)$, there exists a neighborhood $N_{\delta_2}(c)$ in which $f(x)$ is bounded for some $\delta_2 > 0$. If $D - \mathcal{A}$ is nonempty, then $N_{\delta_2}(c) \cap (D - \mathcal{A})$ is also nonempty [if $N_{\delta_2}(c) \subset \mathcal{A}$, then $c + (\delta_2/2) \in \mathcal{A}$, a contradiction]. Let $x_0 \in N_{\delta_2}(c) \cap (D - \mathcal{A})$. Then, on one hand, $f(x_0)$ is not bounded, since $x_0 \in (D - \mathcal{A})$. On the other hand, $f(x_0)$ must be bounded, since $x_0 \in N_{\delta_2}(c)$. This contradiction leads us to conclude that $D - \mathcal{A}$ must be empty and that $f(x)$ is bounded in D . $\square$

Corollary 3.4.1. If $f: D \rightarrow \mathbb{R}$ is continuous, where D is closed and bounded, then $f(x)$ achieves its infimum and supremum at least once in D , that is, there exists $\xi, \eta \in D$ such that

$$\begin{aligned} f(\xi) &\leq f(x) && \text{for all } x \in D, \\ f(\eta) &\geq f(x) && \text{for all } x \in D. \end{aligned}$$

Equivalently,

$$\begin{aligned} f(\xi) &= \inf_{x \in D} f(x), \\ f(\eta) &= \sup_{x \in D} f(x). \end{aligned}$$

Proof. By Theorem 3.4.5, $f(D)$ is a bounded set. Hence, its least upper bound exists. Let $M = \text{lub } f(D)$, which is the same as $\sup_{x \in D} f(x)$. If there exists no point x in D for which $f(x) = M$, then $M - f(x) > 0$ for all $x \in D$. Consequently, $1/[M - f(x)]$ is continuous on D by Theorem 3.4.1, and is hence bounded there by Theorem 3.4.5.

Now, if $\delta > 0$ is any given positive number, we can find a value x for which $f(x) > M - \delta$, or

$$\frac{1}{M - f(x)} > \frac{1}{\delta}.$$

This implies that $1/[M - f(x)]$ is not bounded, a contradiction. Therefore, there must exist a point $\eta \in D$ at which $f(\eta) = M$.

The proof concerning the existence of a point $\xi \in D$ such that $f(\xi) = \inf_{x \in D} f(x)$ is similar. $\square$

The requirement that D be closed in Corollary 3.4.1 is essential. For example, the function $f(x) = 2x - 1$, which is defined on $D = \{x | 0 < x < 1\}$, cannot achieve its infimum, namely -1 , in D . For if there exists a $\xi \in D$ such that $f(\xi) \leq 2x - 1$ for all $x \in D$, then there exists a $\delta > 0$ such that $0 < \xi - \delta$. Hence,

$$f(\xi - \delta) = 2\xi - 2\delta - 1 < f(\xi),$$

a contradiction.

Theorem 3.4.6. Let $f: D \rightarrow R$ be continuous on D . If D is closed and bounded, then f is uniformly continuous on D .

Proof. Suppose that f is not uniformly continuous. Then, by using the logical negation of the statement concerning uniform continuity in Definition 3.4.5, we may conclude that there exists an $\epsilon > 0$ such that for every $\delta > 0$, we can find $x_1, x_2 \in D$ with $|x_1 - x_2| < \delta$ for which $|f(x_1) - f(x_2)| \geq \epsilon$. On this basis, by choosing $\delta = 1$, we can find $u_1, v_1 \in D$ with $|u_1 - v_1| < 1$ for which $|f(u_1) - f(v_1)| \geq \epsilon$. Similarly, we can find $u_2, v_2 \in D$ with $|u_2 - v_2| < \frac{1}{2}$ for which $|f(u_2) - f(v_2)| \geq \epsilon$. By continuing in this process we can find $u_n, v_n \in D$ with $|u_n - v_n| < 1/n$ for which $|f(u_n) - f(v_n)| \geq \epsilon$, $n = 3, 4, \dots$.

Now, let S be the set

$$S = \{u_n | n = 1, 2, \dots\}$$

This set is bounded, since $S \subset D$. Hence, its least upper bound exists. Let $c = \text{lub}(S)$. Since D is closed, then $c \in D$. Thus, as in the proof of Theorem 3.4.4, we can find points $u_{n_1}, u_{n_2}, \dots, u_{n_k}, \dots$ in S such that $\lim_{k \rightarrow \infty} u_{n_k} = c$. Since $f(x)$ is continuous, there exists a $\delta' > 0$ such that

$$|f(x) - f(c)| < \frac{\epsilon}{2},$$

if $|x - c| < \delta'$ for any given $\epsilon > 0$. Let us next choose k large enough such that if $n_k > N$, where N is some large positive number, then

$$|u_{n_k} - c| < \frac{\delta'}{2} \quad \text{and} \quad \frac{1}{n_k} < \frac{\delta'}{2}. \quad (3.16)$$

Since $|u_{n_k} - v_{n_k}| < 1/n_k$, then

$$\begin{aligned} |v_{n_k} - c| &\leq |u_{n_k} - v_{n_k}| + |u_{n_k} - c| \\ &< \frac{1}{n_k} + \frac{\delta'}{2} < \delta' \end{aligned} \quad (3.17)$$

for $n_k > N$. From (3.16) and (3.17) and the continuity of $f(x)$ we conclude that

$$|f(u_{n_k}) - f(c)| < \frac{\epsilon}{2} \quad \text{and} \quad |f(v_{n_k}) - f(c)| < \frac{\epsilon}{2}.$$

Thus,

$$\begin{aligned} |f(u_{n_k}) - f(v_{n_k})| &\leq |f(u_{n_k}) - f(c)| + |f(v_{n_k}) - f(c)| \\ &< \epsilon. \end{aligned} \tag{3.18}$$

However, as was seen earlier,

$$|f(u_n) - f(v_n)| \geq \epsilon,$$

hence,

$$|f(u_{n_k}) - f(v_{n_k})| \geq \epsilon,$$

which contradicts (3.18). This leads us to assert that $f(x)$ is uniformly continuous on D . $\square$

3.4.2. Lipschitz Continuous Functions

Lipschitz continuity is a specialized form of uniform continuity.

Definition 3.4.6. The function $f: D \rightarrow R$ is said to satisfy the Lipschitz condition on a set $E \subset D$ if there exist constants, K and α , where $K > 0$ and $0 < \alpha \leq 1$ such that

$$|f(x_1) - f(x_2)| \leq K|x_1 - x_2|^\alpha$$

for all $x_1, x_2 \in E$. $\square$

Notationally, whenever $f(x)$ satisfies the Lipschitz condition with constants K and α on a set E , we say that it is $\text{Lip}(K, \alpha)$ on E . In this case, $f(x)$ is called a Lipschitz continuous function. It is easy to see that a Lipschitz continuous function on E is also uniformly continuous there.

As an example of a Lipschitz continuous function, consider $f(x) = \sqrt{x}$, where $x \geq 0$. We claim that $\sqrt{x}$ is $\text{Lip}(1, \frac{1}{2})$ on its domain. To show this, we first write

$$|\sqrt{x_1} - \sqrt{x_2}| \leq \sqrt{x_1} + \sqrt{x_2}.$$

Hence,

$$\left| \sqrt{x_1} - \sqrt{x_2} \right|^2 \leq |x_1 - x_2|.$$

Thus,

$$\left| \sqrt{x_1} - \sqrt{x_2} \right| \leq |x_1 - x_2|^{1/2},$$

which proves our claim.

3.5. INVERSE FUNCTIONS

From Chapter 1 we recall that one of the basic characteristics of a function $y = f(x)$ is that two values of y are equal if they correspond to the same value of x . If we were to reverse the roles of x and y so that two values of x are equal whenever they correspond to the same value of y , then x becomes a function of y . Such a function is called the inverse function of f and is denoted by f^{-1} . We conclude that the inverse of $f: D \rightarrow R$ exists if and only if f is one-to-one.

Definition 3.5.1. Let $f: D \rightarrow R$. If there exists a function $f^{-1}: f(D) \rightarrow D$ such that $f^{-1}[f(x)] = x$ and all $x \in D$ and $f[f^{-1}(y)] = y$ for all $y \in f(D)$, then f^{-1} is called the inverse function of f . $\square$

Definition 3.5.2. Let $f: D \rightarrow R$. Then, f is said to be monotone increasing [decreasing] on D if whenever $x_1, x_2 \in D$ are such that $x_1 < x_2$, then $f(x_1) \leq f(x_2)$ [$f(x_1) \geq f(x_2)$]. The function f is strictly monotone increasing [decreasing] on D if $f(x_1) < f(x_2)$ [$f(x_1) > f(x_2)$] whenever $x_1 < x_2$. $\square$

If f is either monotone increasing or monotone decreasing on D , then it is called a monotone function on D . In particular, if it is either strictly monotone increasing or strictly monotone decreasing, then $f(x)$ is strictly monotone on D .

Strictly monotone functions have the property that their inverse functions exist. This will be shown in the next theorem.

Theorem 3.5.1. Let $f: D \rightarrow R$ be strictly monotone increasing (or decreasing) on D . Then, there exists a unique inverse function f^{-1} , which is strictly monotone increasing (or decreasing) on $f(D)$.

Proof. Let us suppose that f is strictly monotone increasing on D . To show that f^{-1} exists as a strictly monotone increasing function on $f(D)$.

Suppose that $x_1, x_2 \in D$ are such that $f(x_1) = f(x_2) = y$. If $x_1 \neq x_2$, then $x_1 < x_2$ or $x_2 < x_1$. Since f is strictly monotone increasing, then $f(x_1) < f(x_2)$ or $f(x_2) < f(x_1)$. In either case, $f(x_1) \neq f(x_2)$, which contradicts the assumption that $f(x_1) = f(x_2)$. Hence, $x_1 = x_2$, that is, f is one-to-one and has therefore a unique inverse f^{-1} .

The inverse f^{-1} is strictly monotone increasing on $f(D)$. To show this, suppose that $f(x_1) < f(x_2)$. Then, $x_1 < x_2$. If not, we must have $x_1 \geq x_2$. In this case, $f(x_1) = f(x_2)$ when $x_1 = x_2$, or $f(x_1) > f(x_2)$ when $x_1 > x_2$, since f is strictly monotone increasing. However, this is contrary to the assumption that $f(x_1) < f(x_2)$. Thus $x_1 < x_2$ and f^{-1} is strictly monotone increasing.

The proof of Theorem 3.5.1 when “increasing” is replaced with “decreasing” is similar. $\square$

Theorem 3.5.2. Suppose that $f: D \rightarrow R$ is continuous and strictly monotone increasing (decreasing) on $[a, b] \subset D$. Then, f^{-1} is continuous and strictly monotone increasing (decreasing) on $f([a, b])$.

Proof. By Theorem 3.5.1 we only need to show the continuity of f^{-1} . Suppose that f is strictly monotone increasing. The proof when f is strictly monotone decreasing is similar.

Since f is continuous on a closed and bounded interval, then by Corollary 3.4.1 it must achieve its infimum and supremum on $[a, b]$. Furthermore, because f is strictly monotone increasing, its infimum and supremum must be attained at only a and b , respectively. Thus

$$f([a, b]) = [f(a), f(b)].$$

Let $d \in [f(a), f(b)]$. There exists a unique value c , $a \leq c \leq b$, such that $f(c) = d$. For any $\epsilon > 0$, let τ be defined as

$$\tau = \min[f(c) - f(c - \epsilon), f(c + \epsilon) - f(c)].$$

Then there exists a δ , $0 < \delta < \tau$, such that all the x 's in $[a, b]$ that satisfy the inequality

$$|f(x) - d| < \delta$$

must also satisfy the inequality

$$|x - c| < \epsilon.$$

This is true because

$$\begin{aligned} f(c) - f(c) + f(c - \epsilon) &< d - \delta < f(x) < d + \delta \\ &< f(c) + f(c + \epsilon) - f(c), \end{aligned}$$

that is,

$$f(c - \epsilon) < f(x) < f(c + \epsilon).$$

Using the fact that f^{-1} is strictly monotone increasing (by Theorem 3.5.1), we conclude that

$$c - \epsilon < x < c + \epsilon,$$

that is, $|x - c| < \epsilon$. It follows that $x = f^{-1}(y)$ is continuous on $[f(a), f(b)]$. $\square$

Note that in general if $y = f(x)$, the equation $y - f(x) = 0$ may not produce a unique solution for x in terms of y . If, however, the domain of f can be partitioned into subdomains on each of which f is strictly monotone, then f can have an inverse on each of these subdomains.

EXAMPLE 3.5.1. Consider the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $y = f(x) = x^3$. It is easy to see that f is strictly monotone increasing for all $x \in \mathbb{R}$. It therefore has a unique inverse given by $f^{-1}(y) = y^{1/3}$.

EXAMPLE 3.5.2. Let $f: [-1, 1] \rightarrow \mathbb{R}$ be such that $y = f(x) = x^5 - x$. From Figure 3.2 it can be seen that f is strictly monotone increasing on $D_1 = [-1, -5^{-1/4}]$ and $D_2 = [5^{-1/4}, 1]$, but is strictly monotone decreasing on $D_3 = [-5^{-1/4}, 5^{-1/4}]$. This function has therefore three inverses, one on each of D_1 , D_2 , and D_3 . By Theorem 3.5.2, all three inverses are continuous.

EXAMPLE 3.5.3. Let $f: \mathbb{R} \rightarrow [-1, 1]$ be the function $y = f(x) = \sin x$, where x is measured in radians. There is no unique inverse on $\mathbb{R}$, since the sine function is not strictly monotone there. If, however, we restrict the domain of f to $D = [-\pi/2, \pi/2]$, then f is strictly monotone increasing there and has the unique inverse $f^{-1}(y) = \text{Arcsin } y$ (see Example 1.3.4). The inverse of f on $[\pi/2, 3\pi/2]$ is given by $f^{-1}(y) = \pi - \text{Arcsin } y$. We can similarly find the inverse of f on $[3\pi/2, 5\pi/2]$, $[5\pi/2, 7\pi/2]$, etc.

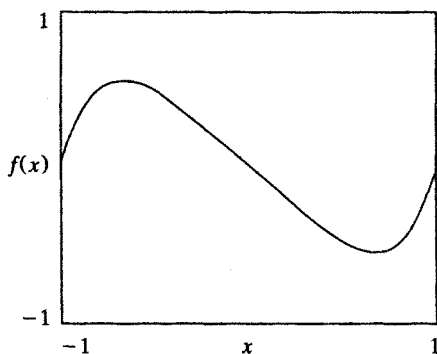


Figure 3.2. The graph of the function $f(x) = x^5 - x$.

3.6. CONVEX FUNCTIONS

Convex functions are frequently used in operations research. They also happen to be continuous, as will be shown in this section. The natural domains for such functions are convex sets.

Definition 3.6.1. A set $D \subset R$ is convex if $\lambda x_1 + (1 - \lambda)x_2 \in D$ whenever x_1, x_2 belong to D and $0 \leq \lambda \leq 1$. Geometrically, a convex set contains the line segment connecting any two of its points. The same definition actually applies to convex sets in R^n , the n -dimensional Euclidean space ($n \geq 2$). For example, each of the following sets is convex:

1. Any interval in R .
2. Any sphere in R^3 , and in general, any hypersphere in R^n , $n \geq 4$.
3. The set $\{(x, y) \in R^2 \mid |x| + |y| \leq 1\}$. See Figure 3.3. $\square$

Definition 3.6.2. A function $f: D \rightarrow R$ is convex if

$$f[\lambda x_1 + (1 - \lambda)x_2] \leq \lambda f(x_1) + (1 - \lambda)f(x_2) \quad (3.19)$$

for all $x_1, x_2 \in D$ and any λ such that $0 \leq \lambda \leq 1$. The function f is strictly convex if inequality (3.19) is strict for $x_1 \neq x_2$.

Geometrically, inequality (3.19) means that if P and Q are any two points on the graph of $y = f(x)$, then the portion of the graph between P and Q lies below the chord PQ (see Figure 3.4). Examples of convex functions include $f(x) = x^2$ on R , $f(x) = \sin x$ on $[\pi, 2\pi]$, $f(x) = e^x$ on R , $f(x) = -\log x$ for $x > 0$, to name just a few. $\square$

Definition 3.6.3. A function $f: D \rightarrow R$ is concave if $-f$ is convex. $\square$

We note that if $f: [a, b] \rightarrow R$ is convex and the values of f at a and b are finite, then $f(x)$ is bounded from above on $[a, b]$ by $M = \max\{f(a), f(b)\}$. This is true because if $x \in [a, b]$, then $x = \lambda a + (1 - \lambda)b$ for some $\lambda \in [0, 1]$,

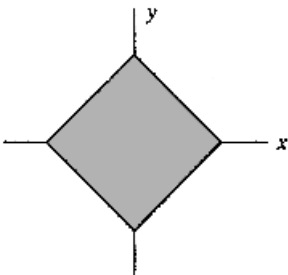


Figure 3.3. The set $\{(x, y) \in R^2 \mid |x| + |y| \leq 1\}$.

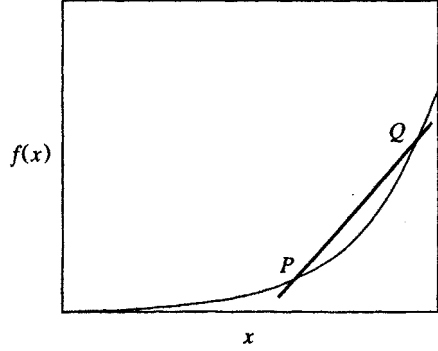


Figure 3.4. The graph of a convex function.

since $[a, b]$ is a convex set. Hence,

$$\begin{aligned} f(x) &\leq \lambda f(a) + (1 - \lambda)f(b) \\ &\leq \lambda M + (1 - \lambda)M = M. \end{aligned}$$

The function $f(x)$ is also bounded from below. To show this, we first note that any $x \in [a, b]$ can be written as

$$x = \frac{a+b}{2} + t,$$

where

$$a - \frac{a+b}{2} \leq t \leq b - \frac{a+b}{2}.$$

Now,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}f\left(\frac{a+b}{2} + t\right) + \frac{1}{2}f\left(\frac{a+b}{2} - t\right), \quad (3.20)$$

since if $(a+b)/2 + t$ belongs to $[a, b]$, then so does $(a+b)/2 - t$. From (3.20) we then have

$$f\left(\frac{a+b}{2} + t\right) \geq 2f\left(\frac{a+b}{2}\right) - f\left(\frac{a+b}{2} - t\right).$$

Since

$$f\left(\frac{a+b}{2} - t\right) \leq M,$$

then

$$f\left(\frac{a+b}{2} + t\right) \geq 2f\left(\frac{a+b}{2}\right) - M,$$

that is, $f(x) \geq m$ for all $x \in [a, b]$, where

$$m = 2f\left(\frac{a+b}{2}\right) - M.$$

Another interesting property of convex functions is given in Theorem 3.6.1.

Theorem 3.6.1. Let $f: D \rightarrow \mathbb{R}$ be a convex function, where D is an open interval. Then f is $\text{Lip}(K, 1)$ on any closed interval $[a, b]$ contained in D , that is,

$$|f(x_1) - f(x_2)| \leq K|x_1 - x_2| \quad (3.21)$$

for all $x_1, x_2 \in [a, b]$.

Proof. Consider the closed interval $[a - \epsilon, b + \epsilon]$, where $\epsilon > 0$ is chosen so that this interval is contained in D . Let m' and M' be, respectively, the lower and upper bounds of f (as was seen earlier) on $[a - \epsilon, b + \epsilon]$. Let x_1, x_2 be any two distinct points in $[a, b]$. Define z_1 and λ as

$$z_1 = x_2 + \frac{\epsilon(x_2 - x_1)}{|x_1 - x_2|},$$

$$\lambda = \frac{|x_1 - x_2|}{\epsilon + |x_1 - x_2|}.$$

Then $z_1 \in [a - \epsilon, b + \epsilon]$. This is true because $(x_1 - x_1)/|x_1 - x_2|$ is either equal to 1 or to -1 . Since $x_2 \in [a, b]$, then

$$a - \epsilon \leq x_2 - \epsilon \leq x_2 + \frac{\epsilon(x_2 - x_1)}{|x_1 - x_2|} \leq x_2 + \epsilon \leq b + \epsilon.$$

Furthermore, it can be verified that

$$x_2 = \lambda z_1 + (1 - \lambda)x_1.$$

We then have

$$f(x_2) \leq \lambda f(z_1) + (1 - \lambda)f(x_1) = \lambda[f(z_1) - f(x_1)] + f(x_1).$$

Thus,

$$\begin{aligned} f(x_2) - f(x_1) &\leq \lambda[f(z_1) - f(x_1)] \\ &\leq \lambda[M' - m'] \\ &\leq \frac{|x_1 - x_2|}{\epsilon}(M' - m') = K|x_1 - x_2|, \end{aligned} \quad (3.22)$$

where $K = (M' - m')/\epsilon$. Since inequality (3.22) is true for any $x_1, x_2 \in [a, b]$, we must also have

$$f(x_1) - f(x_2) \leq K|x_1 - x_2|. \quad (3.23)$$

From inequalities (3.22) and (3.23) we conclude that

$$|f(x_1) - f(x_2)| \leq K|x_1 - x_2|$$

for any $x_1, x_2 \in [a, b]$, which shows that $f(x)$ is $\text{Lip}(K, 1)$ on $[a, b]$. $\square$

Using Theorem 3.6.1 it is easy to prove the following corollary:

Corollary 3.6.1. Let $f: D \rightarrow R$ be a convex function, where D is an open interval. If $[a, b]$ is any closed interval contained in D , then $f(x)$ is uniformly continuous on $[a, b]$ and is therefore continuous on D .

Note that if $f(x)$ is convex on (a, b) , then it does not have to be continuous at the end points of the interval. It is easy to see, for example, that the function $f: [-1, 1] \rightarrow R$ defined as

$$f(x) = \begin{cases} x^2, & -1 < x < 1, \\ 2, & x = 1, -1 \end{cases}$$

is convex on $(-1, 1)$, but is discontinuous at $x = -1, 1$.

3.7. CONTINUOUS AND CONVEX FUNCTIONS IN STATISTICS

The most vivid examples of continuous functions in statistics are perhaps the cumulative distribution functions of continuous random variables. If X is a continuous random variable, then its cumulative distribution function

$$F(x) = P(X \leq x)$$

is continuous on R . In this case,

$$P(X = a) = \lim_{n \rightarrow \infty} \left[F\left(a + \frac{1}{n}\right) - F\left(a - \frac{1}{n}\right) \right] = 0,$$

that is, the distribution of X assigns a zero probability to any single value. This is a basic characteristic of continuous random variables.

Examples of continuous distributions include the beta, Cauchy, chi-squared, exponential, gamma, Laplace, logistic, lognormal, normal, t , uniform, and the Weibull distributions. Most of these distributions are described

in introductory statistics books. A detailed account of their properties and uses is given in the two books by Johnson and Kotz (1970a, 1970b).

It is interesting to note that if X is any random variable (not necessarily continuous), then its cumulative distribution function, $F(x)$, is right-continuous on R (see, for example, Harris, 1966, page 55). This function is also monotone increasing on R . If $F(x)$ is strictly monotone increasing, then by Theorem 3.5.1 it has a unique inverse $F^{-1}(y)$. In this case, if Y has the uniform distribution over the open interval $(0, 1)$, then the random variable $F^{-1}(Y)$ has the cumulative distribution function $F(x)$. To show this, consider $X = F^{-1}(Y)$. Then,

$$\begin{aligned} P(X \leq x) &= P[F^{-1}(Y) \leq x] \\ &= P[Y \leq F(x)] \\ &= F(x). \end{aligned}$$

This result has an interesting application in sampling. If $Y_1, Y_2, \dots, Y_n$ form an independent random sample from the uniform distribution $U(0, 1)$, then $F^{-1}(Y_1), F^{-1}(Y_2), \dots, F^{-1}(Y_n)$ will form an independent sample from a distribution with the cumulative distribution function $F(x)$. In other words, samples from any distribution can be generated through sampling from the uniform distribution $U(0, 1)$. This result forms the cornerstone of Monte Carlo simulation in statistics. Such a method provides an artificial way of collecting "data." There are situations where the actual taking of a physical sample is either impossible or too expensive. In such situations, useful information can often be derived from simulated sampling. Monte Carlo simulation is also used in the study of the relative performance of test statistics and parameter estimators when the data come from certain specified parent distributions.

Another example of the use of continuous functions in statistics is in limit theory. For example, it is known that if $\{X_n\}_{n=1}^{\infty}$ is a sequence of random variables that converges in probability to c , and if $g(x)$ is a continuous function at $x = c$, then the random variable $g(X_n)$ converges in probability to $g(c)$ as $n \rightarrow \infty$. By definition, a sequence of random variables $\{X_n\}_{n=1}^{\infty}$ converges in probability to a constant c if for a given $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} P(|X_n - c| \geq \epsilon) = 0.$$

In particular, if $\{X_n\}_{n=1}^{\infty}$ is a sequence of estimators of a parameter c , then X_n is said to be a consistent estimator of c if X_n converges in probability to c . For example, the sample mean

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$$

of a sample of size n from a population with a finite mean μ is a consistent estimator of μ according to the law of large numbers (see, for example, Lindgren, 1976, page 155). Other types of convergence in statistics can be found in standard mathematical statistics books (see the annotated bibliography).

Convex functions also play an important role in statistics, as can be seen from the following examples.

If $f(x)$ is a convex function and X is a random variable with a finite mean $\mu = E(X)$, then

$$E[f(X)] \geq f[E(X)].$$

Equality holds if and only if X is constant with probability 1. This inequality is known as *Jensen's inequality*. If f is strictly convex, the inequality is strict unless X is constant with probability 1. A proof of Jensen's inequality is given in Section 6.7.4. See also Lehmann (1983, page 50).

Jensen's inequality has useful applications in statistics. For example, it can be used to show that if $x_1, x_2, \dots, x_n$ are n positive scalars, then their arithmetic mean is greater than or equal to their geometric mean, which is equal to $(\prod_{i=1}^n x_i)^{1/n}$. This can be shown as follows:

Consider the convex function $f(x) = -\log x$. Let X be a discrete random variable that takes the values $x_1, x_2, \dots, x_n$ with probabilities equal to $1/n$, that is,

$$P(X=x) = \begin{cases} \frac{1}{n}, & x = x_1, x_2, \dots, x_n \\ 0 & \text{otherwise.} \end{cases}$$

Then, by Jensen's inequality,

$$E(-\log X) \geq -\log E(X). \quad (3.24)$$

However,

$$E(-\log X) = -\frac{1}{n} \sum_{i=1}^n \log x_i, \quad (3.25)$$

and

$$-\log E(X) = -\log \left(\frac{1}{n} \sum_{i=1}^n x_i \right). \quad (3.26)$$

By using (3.25) and (3.26) in (3.24) we get

$$\frac{1}{n} \sum_{i=1}^n \log x_i \leq \log \left(\frac{1}{n} \sum_{i=1}^n x_i \right),$$

or

$$\log \left[\left(\prod_{i=1}^n x_i \right)^{1/n} \right] \leq \log \left(\frac{1}{n} \sum_{i=1}^n x_i \right).$$

Since the logarithmic function is monotone increasing, we conclude that

$$\left(\prod_{i=1}^n x_i \right)^{1/n} \leq \frac{1}{n} \sum_{i=1}^n x_i.$$

Jensen's inequality can also be used to show that the arithmetic mean is greater than or equal to the harmonic mean. This assertion can be shown as follows:

Consider the function $f(x) = x^{-1}$, which is convex for $x > 0$. If X is a random variable with $P(X > 0) = 1$, then by Jensen's inequality,

$$E\left(\frac{1}{X}\right) \geq \frac{1}{E(X)}. \quad (3.27)$$

In particular, if X has the discrete distribution described earlier, then

$$E\left(\frac{1}{X}\right) = \frac{1}{n} \sum_{i=1}^n \frac{1}{x_i}$$

and

$$E(X) = \frac{1}{n} \sum_{i=1}^n x_i.$$

By substitution in (3.27) we get

$$\frac{1}{n} \sum_{i=1}^n \frac{1}{x_i} \geq \left(\frac{1}{n} \sum_{i=1}^n x_i \right)^{-1},$$

or

$$\frac{1}{n} \sum_{i=1}^n x_i \geq \left(\frac{1}{n} \sum_{i=1}^n \frac{1}{x_i} \right)^{-1}. \quad (3.28)$$

The quantity on the right of inequality (3.28) is the harmonic mean of $x_1, x_2, \dots, x_n$.

Another example of the use of convex functions in statistics is in the general theory of estimation. Let $X_1, X_2, \dots, X_n$ be a random sample of size n from a population whose distribution depends on an unknown parameter θ . Let $\omega(X_1, X_2, \dots, X_n)$ be an estimator of θ . By definition, the loss function $L[\theta, \omega(X_1, X_2, \dots, X_n)]$ is a nonnegative function that measures the loss incurred when θ is estimated by $\omega(X_1, X_2, \dots, X_n)$. The expected value (mean) of the loss function is called the risk function, denoted by $R(\theta, \omega)$, that is,

$$R(\theta, \omega) = E\{L[\theta, \omega(X_1, X_2, \dots, X_n)]\}.$$

The loss function is taken to be a convex function of θ . Examples of loss functions include the squared error loss,

$$L[\theta, \omega(X_1, X_2, \dots, X_n)] = [\theta - \omega(X_1, X_2, \dots, X_n)]^2,$$

and the absolute error loss,

$$L[\theta, \omega(X_1, X_2, \dots, X_n)] = |\theta - \omega(X_1, X_2, \dots, X_n)|.$$

The first loss function is strictly convex, whereas the second is convex, but not strictly convex.

The goodness of an estimator of θ is judged on the basis of its risk function, assuming that a certain loss function has been selected. The smaller the risk, the more desirable the estimator. An estimator $\omega^*(X_1, X_2, \dots, X_n)$ is said to be admissible if there is no other estimator $\omega(X_1, X_2, \dots, X_n)$ of θ such that

$$R(\theta, \omega) \leq R(\theta, \omega^*)$$

for all $\theta \in \Omega$ (Ω is the parameter space), with strict inequality for at least one θ . An estimator $\omega_0(X_1, X_2, \dots, X_n)$ is said to be a minimax estimator if it minimizes the supremum (with respect to θ) of the risk function, that is,

$$\sup_{\theta \in \Omega} R(\theta, \omega_0) \leq \sup_{\theta \in \Omega} R(\theta, \omega),$$

where $\omega(X_1, X_2, \dots, X_n)$ is any other estimator of θ . It should be noted that a minimax estimator may not be admissible.

EXAMPLE 3.7.1. Let $X_1, X_2, \dots, X_{20}$ be a random sample of size 20 from the normal distribution $N(\theta, 1)$, $-\infty < \theta < \infty$. Let $\omega_1(X_1, X_2, \dots, X_{20}) = \bar{X}_{20}$ be the sample mean, and let $\omega_2(X_1, X_2, \dots, X_{20}) = 0$. Then, using a squared

error loss function,

$$R(\theta, \omega_1) = E\left[(\bar{X}_{20} - \theta)^2\right] = \text{Var}(\bar{X}_{20}) = \frac{1}{20},$$

$$R(\theta, \omega_2) = E\left[(0 - \theta)^2\right] = \theta^2.$$

In this case,

$$\sup_{\theta} [R(\theta, \omega_1)] = \frac{1}{20},$$

whereas

$$\sup_{\theta} [R(\theta, \omega_2)] = \infty.$$

Thus $\omega_1 = \bar{X}_{20}$ is a better estimator than $\omega_2 = 0$. It can be shown that $\bar{X}_{20}$ is the minimax estimator of θ with respect to a squared error loss function. Note, however, that $\bar{X}_{20}$ is not an admissible estimator, since

$$R(\theta, \omega_1) \leq R(\theta, \omega_2)$$

for $\theta \geq 20^{-1/2}$ or $\theta \leq -20^{-1/2}$. However, for $-20^{-1/2} < \theta < 20^{-1/2}$,

$$R(\theta, \omega_2) < R(\theta, \omega_1).$$

FURTHER READING AND ANNOTATED BIBLIOGRAPHY

- Corwin, L. J., and R. H. Szczarba (1982). *Multivariable Calculus*. Dekker, New York.
- Fisz, M. (1963). *Probability Theory and Mathematical Statistics*, 3rd ed. Wiley, New York. (Some continuous distributions are described in Chap. 5. Limit theorems concerning sequences of random variables are discussed in Chap. 6.)
- Fulks, W. (1978). *Advanced Calculus*, 3rd ed. Wiley, New York.
- Hardy, G. H. (1955). *A Course of Pure Mathematics*, 10th ed. The University Press, Cambridge, England. (Section 89 of Chap. 4 gives definitions concerning the o and O symbols introduced in Section 3.3 of this chapter.)
- Harris, B. (1966). *Theory of Probability*. Addison-Wesley, Reading, Massachusetts. (Some continuous distributions are given in Section 3.5.)
- Henle, J. M., and E. M. Kleinberg (1979). *Infinitesimal Calculus*. The MIT Press, Cambridge, Massachusetts.
- Hillier, F. S., and G. J. Lieberman (1967). *Introduction to Operations Research*. Holden-Day, San Francisco. (Convex sets and functions are described in Appendix 1.)
- Hogg, R. V., and A. T. Craig (1965). *Introduction to Mathematical Statistics*, 2nd ed. Macmillan, New York. (Loss and risk functions are discussed in Section 9.3.)
- Hyslop, J. M. (1954). *Infinite Series*, 5th ed. Oliver and Boyd, Edinburgh, England. (Chap. 1 gives definitions and summaries of results concerning the o, O notation.)

- Johnson, N. L., and S. Kotz (1970a). *Continuous Univariate Distributions—1*. Houghton Mifflin, Boston.
- Johnson, N. L., and S. Kotz (1970b). *Continuous Univariate Distributions—2*. Houghton Mifflin, Boston.
- Lehmann, E. L. (1983). *Theory of Point Estimation*. Wiley, New York. (Section 1.6 discusses convex functions and their uses as loss functions.)
- Lindgren, B. W. (1976). *Statistical Theory*, 3rd ed. Macmillan, New York. (The concepts of loss and utility, or negative loss, used in statistical decision theory are discussed in Chap. 8.)
- Randles, R. H., and D. A. Wolfe (1979). *Introduction to the Theory of Nonparametric Statistics*. Wiley, New York. (Some mathematical statistics results, including Jensen's inequality, are given in the Appendix.)
- Rao, C. R. (1973). *Linear Statistical Inference and Its Applications*, 2nd ed. Wiley, New York.
- Roberts, A. W., and D. E. Varberg (1973). *Convex Functions*. Academic Press, New York. (This is a handy reference book that contains all the central facts about convex functions.)
- Roussas, G. G. (1973). *A First Course in Mathematical Statistics*. Addison-Wesley, Reading, Massachusetts. (Chap. 12 defines and provides discussions concerning admissible and minimax estimators.)
- Rudin, W. (1964). *Principles of Mathematical Analysis*, 2nd ed. McGraw-Hill, New York. (Limits of functions and some properties of continuous functions are given in Chap. 4.)
- Sagan, H. (1974). *Advanced Calculus*. Houghton Mifflin, Boston.
- Smith, D. E. (1958). *History of Mathematics*, Vol. 1. Dover, New York.

EXERCISES

In Mathematics

3.1. Determine if the following limits exist:

$$(a) \quad \lim_{x \rightarrow 1} \frac{x^5 - 1}{x - 1},$$

$$(b) \quad \lim_{x \rightarrow 0} x \sin \frac{1}{x},$$

$$(c) \quad \lim_{x \rightarrow 0} \left(\sin \frac{1}{x} \right) / \left(\sin \frac{1}{x} \right),$$

$$(d) \quad \lim_{x \rightarrow 0} f(x), \text{ where } f(x) = \begin{cases} \frac{x^2 - 1}{x - 1}, & x > 0, \\ \frac{x^3 - 1}{2(x - 1)}, & x < 0. \end{cases}$$

3.2. Show that

(a) $\tan x^3 = o(x^2)$ as $x \rightarrow 0$.

(b) $x = o(\sqrt{x})$ as $x \rightarrow 0$,

(c) $O(1) = o(x)$ as $x \rightarrow \infty$,

(d) $f(x)g(x) = x^{-1} + O(1)$ as $x \rightarrow 0$, where $f(x) = x + o(x^2)$, $g(x) = x^{-2} + O(x^{-1})$.

3.3. Determine where the following functions are continuous, and indicate the points of discontinuity (if any):

(a)
$$f(x) = \begin{cases} x \sin(1/x), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

(b)
$$f(x) = \begin{cases} [(x-1)/(2-x)]^{1/2}, & x \neq 2, \\ 1, & x = 2, \end{cases}$$

(c)
$$f(x) = \begin{cases} x^{-m/n}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

where m and n are positive integers,

(d)
$$f(x) = \frac{x^4 - 2x^2 + 3}{x^3 - 1}, \quad x \neq 1.$$

3.4. Show that $f(x)$ is continuous at $x = a$ if and only if it is both left-continuous and right-continuous at $x = a$.

3.5. Use Definition 3.4.1 to show that the function

$$f(x) = x^2 - 1$$

is continuous at any point $a \in \mathbb{R}$.

3.6. For what values of x is

$$f(x) = \lim_{n \rightarrow \infty} \frac{3nx}{1 - nx}$$

continuous?

3.7. Consider the function

$$f(x) = \frac{x - |x|}{x}, \quad -1 < x < 1, \quad x \neq 0.$$

Can $f(x)$ be defined at $x = 0$ so that it will be continuous there?

3.8. Let $f(x)$ be defined for all $x \in \mathbb{R}$ and continuous at $x = 0$. Furthermore,

$$f(a + b) = f(a) + f(b),$$

for all a, b in $\mathbb{R}$. Show that $f(x)$ is uniformly continuous everywhere in $\mathbb{R}$.

3.9. Let $f(x)$ be defined as

$$f(x) = \begin{cases} 2x - 1, & 0 \leq x \leq 1, \\ x^3 - 5x^2 + 5, & 1 \leq x \leq 2. \end{cases}$$

Determine if $f(x)$ is uniformly continuous on $[0, 2]$.

3.10. Show that $f(x) = \cos x$ is uniformly continuous on $\mathbb{R}$.

3.11. Prove Theorem 3.4.1.

3.12. A function $f: D \rightarrow \mathbb{R}$ is called upper semicontinuous at $a \in D$ if for a given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$f(x) < f(a) + \epsilon$$

for all $x \in N_\delta(a) \cap D$. If the above inequality is replaced with

$$f(x) > f(a) - \epsilon,$$

then $f(x)$ is said to be lower semicontinuous.

Show that if D is closed and bounded, then

(a) $f(x)$ is bounded from above on D if $f(x)$ is upper semicontinuous.

(b) $f(x)$ is bounded from below on D if $f(x)$ is lower semicontinuous.

3.13. Let $f: [a, b] \rightarrow \mathbb{R}$ be continuous such that $f(x) = 0$ for every rational number in $[a, b]$. Show that $f(x) = 0$ for every x in $[a, b]$.

3.14. For what values of x does the function

$$f(x) = 3 + |x - 1| + |x + 1|$$

have a unique inverse?

3.15. Let $f: R \rightarrow R$ be defined as

$$f(x) = \begin{cases} x, & x \leq 1, \\ 2x - 1, & x > 1. \end{cases}$$

Find the inverse function f^{-1} ,

3.16. Let $f(x) = 2x^2 - 8x + 8$. Find the inverse of $f(x)$ for

(a) $x \leq -2$,

(b) $x > 2$.

3.17. Suppose that $f: [a, b] \rightarrow R$ is a convex function. Show that for a given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\sum_{i=1}^n |f(a_i) - f(b_i)| < \epsilon$$

for every finite, pairwise disjoint family of open subintervals $\{(a_i, b_i)\}_{i=1}^n$ of $[a, b]$ for which $\sum_{i=1}^n (b_i - a_i) < \delta$.

Note: A function satisfying this property is said to be absolutely continuous on $[a, b]$.

3.18. Let $f: [a, b] \rightarrow R$ be a convex function. Show that if $a_1, a_2, \dots, a_n$ are positive numbers and $x_1, x_2, \dots, x_n$ are points in $[a, b]$, then

$$f\left(\frac{\sum_{i=1}^n a_i x_i}{A}\right) \leq \frac{\sum_{i=1}^n a_i f(x_i)}{A},$$

where $A = \sum_{i=1}^n a_i$.

3.19. Let $f(x)$ be continuous on $D \subset R$. Let S be the set of all $x \in D$ such that $f(x) = 0$. Show that S is a closed set.

3.20. Let $f(x)$ be a convex function on $D \subset R$. Show that $\exp[f(x)]$ is also convex on D .

In Statistics

3.21. Let X be a continuous random variable with the cumulative distribution function

$$F(x) = 1 - e^{-x/\theta}, \quad x > 0.$$

This is known as the exponential distribution. Its mean and variance are $\mu = \theta$, $\sigma^2 = \theta^2$, respectively. Generate a random sample of five observations from an exponential distribution with mean 2.

[*Hint:* Select a ten-digit number from the table of random numbers, for example, 8389611097. Divide it by 10^{10} to obtain the decimal number 0.8389611097. This number can be regarded as an observation from the uniform distribution $U(0, 1)$. Now, solve the equation $F(x) = 0.8389611097$. The resulting value of x is considered as an observation from the prescribed exponential distribution. Repeat this process four more times, each time selecting a new decimal number from the table of random numbers.]

3.22. Verify Jensen's inequality in each of the following two cases:

- (a) X is normally distributed and $f(x) = |x|$.
- (b) X has the exponential distribution and $f(x) = e^{-x}$.

3.23. Use the definition of convergence in probability to verify that if the sequence of random variables $\{X_n\}_{n=1}^{\infty}$ converges in probability to zero, then so does the sequence $\{X_n^2\}_{n=1}^{\infty}$.

3.24. Show that

$$E(X^2) \geq [E(|X|)]^2.$$

[*Hint:* Let $Y = |X|$. Apply Jensen's inequality to Y with $f(x) = x^2$.] Deduce that if X has a mean μ and a variance σ^2 , then

$$E(|X - \mu|) \leq \sigma.$$

3.25. Consider the exponential distribution described in Exercise 3.21. Let $X_1, X_2, \dots, X_n$ be a sample of size n from this distribution. Consider the following estimators of θ :

- (a) $\omega_1(X_1, X_2, \dots, X_n) = \bar{X}_n$, the sample mean.
- (b) $\omega_2(X_1, X_2, \dots, X_n) = \bar{X}_n + 1$,
- (c) $\omega_3(X_1, X_2, \dots, X_n) = X_n$.

Determine the risk function corresponding to a squared error loss function for each one of these estimators. Which estimator has the smallest risk for all values of θ ?